

Optimizing Rayleigh quotient with symmetric constraints and its application to eigenvalue backward errors of polynomial and rational eigenvalue problems

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Abstract

Let $H \in \mathbb{C}^{n,n}$ be Hermitian and $S_0, S_1, \dots, S_k \in \mathbb{C}^{n,n}$ be symmetric matrices. We consider the problem of maximizing the Rayleigh quotient of H with respect to certain constraints involving symmetric matrices S_0, S_1, \dots, S_k . More precisely, we compute

$$m_{hs_0s_1\dots s_k}(H, S_0, S_1, \dots, S_k) := \sup \left\{ \frac{v^* H v}{v^* v} : v \in \mathbb{C}^n \setminus \{0\}, v^T S_i v = 0 \right. \\ \left. \text{for } i = 0, \dots, k \right\}, \quad (\mathcal{P})$$

where T and $*$ denote respectively the transpose and the conjugate transpose of a matrix or a vector.

Such problems occur in stability analysis of uncertain systems and in the eigenvalue perturbation theory of matrices and matrix polynomials [3, 4]. A particular case of problem (\mathcal{P}) with only one symmetric constraint (i.e., when $k = 0$) is used to characterize the μ -value of the matrix under skew-symmetric perturbations [4]. An explicit computable formula was obtained for $m_{hs_0}(H, S_0)$ in [4, Theorem 6.3] and given by

$$m_{hs_0}(H, S_0) = \inf_{t \in [0, \infty)} \lambda_2 \left(\begin{bmatrix} H & t\bar{S}_0 \\ tS_0 & \bar{H} \end{bmatrix} \right),$$

where $\lambda_2(A)$ stands for the second largest eigenvalue of a Hermitian matrix A . However, the solution to the problem (\mathcal{P}) with more than one symmetric constraint was not known.

We derive an explicit computable formula for (\mathcal{P}) in terms of minimizing the second largest eigenvalue of a parameter-depending Hermitian matrix under a simplicity assumption. The results are then applied to derive computable formulas for the structured eigenvalue backward errors of rational matrix functions (RMFs) of the following form

$$G(z) = A_0 + zA_1 + \dots + z^d A_d + \frac{s_1(z)}{q_1(z)} E_1 + \dots + \frac{s_k(z)}{q_k(z)} E_k$$

where the coefficients $A_p, p = 0, 1, \dots, d$ and $E_j, j = 1, 2, \dots, k$ are $n \times n$ matrices, and $s_j(z), q_j(z)$, for $j = 1, 2, \dots, k$ are scalar polynomials.

Eigenvalue backward errors of matrix polynomials, both for unstructured and structure-preserving perturbations, have been studied in the literature; see [9] for unstructured, [1] for Hermitian and related structures, and [2] for palindromic and related structures. However, the literature on RMFs is relatively limited, and the structured eigenvalue backward errors have not been explored before.

To explore this, we first aim to reformulate the problem of computing structured eigenvalue backward errors for RMFs with symmetric, skew-symmetric, T-even, T-odd, and T-palindromic structures into the optimization problem (\mathcal{P}) . We then apply the results obtained for this optimization

problem to derive computable formulas for the structured eigenvalue backward errors of RMFs. As a specific case of RMFs, formulas for the structured eigenvalue backward errors of matrix polynomials with the aforementioned structures can also be derived. Numerical experiments suggest that our results [5] provide a more accurate estimation of the supremum in (\mathcal{P}) compared to the one in [7]. Some of these results are published in *Linear Algebra and its Applications* [5], while others in *BIT Numerical Mathematics* [6].

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