# Solving Generalized Lyapunov Equations with guarantees: application to the Reduction of Linear Switched Systems.

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Abstract

We deal with the efficient and certified approximation of the *generalized Lyapunov equation* (GLEs)

$$\boldsymbol{A}\boldsymbol{X} + \boldsymbol{X}\boldsymbol{A}^{\mathsf{T}} + \sum_{j=1}^{M}\left(\boldsymbol{N}_j\boldsymbol{X}\boldsymbol{N}_j^{\mathsf{T}}\right) + \boldsymbol{B}\boldsymbol{B}^{\mathsf{T}} = \boldsymbol{0}, \tag{1}$$

where $\boldsymbol{A}, \boldsymbol{N}_j \in \mathbb{R}^{n\times n}$, $\boldsymbol{A}$ is Hurwitz, i.e., its spectrum is contained in the open left-half complex plane, and $\boldsymbol{B} \in \mathbb{R}^{n\times m}$ with $m$ typically much smaller than $n$. GLEs with these features naturally arise in the context of *model order reduction* (MOR) of bilinear control systems [2, 5] and linear parameter-varying systems as well as in the context of stochastic differential equations for stability analysis [4]. For switched linear systems of the form

$$\Sigma_q \quad \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_{q(t)}\boldsymbol{x}(t) + \boldsymbol{B}_{q(t)}\boldsymbol{u}(t), \quad \boldsymbol{x}(t_0) = \boldsymbol{0}, \\ \boldsymbol{y}(t) = \boldsymbol{C}_{q(t)}\boldsymbol{x}(t), \end{cases} \tag{2}$$

the authors of [6] introduced a balancing-based MOR method that requires the solution of certain GLEs. In (2), $q\colon \mathbb{R} \to \mathcal{J} := \{1, \ldots, M\}$ is the external switching signal, which we assume to be an element of the set of allowed switching signals

$$\mathcal{S} := \{q\colon \mathbb{R} \to \mathcal{J} \mid q \text{ is right continuous with locally finite number of jumps}\}. \tag{3}$$

The symbols $\boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{u}(t) \in \mathbb{R}^m$, and $\boldsymbol{y}(t) \in \mathbb{R}^p$ denote the *state*, the controlled *input*, and the measured *output*, respectively. The system matrices $\boldsymbol{A}_j \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}_j \in \mathbb{R}^{n\times m}$, and $\boldsymbol{C}_j \in \mathbb{R}^{p\times n}$ correspond to the *ordinary differential equation* (ODE) active in mode $j \in \mathcal{J}$. Typically one refers to (2) as the *full-order model* (FOM). Sample applications of switched systems include robot manipulators, traffic management, automatic gear shifting, and power systems; see for instance [3] and the references therein.

If (2) has to be evaluated repeatedly, for instance, in a simulation context for different inputs or switching signals, or if matrix equalities or inequalities in the context of synthesis have to be solved, then a large dimension $n$ of the state renders this a computationally expensive task. In such scenarios, one can rely on MOR and replace (2) by the *reduced-order model* (ROM)

$$\tilde{\Sigma}_q \quad \begin{cases} \dot{\tilde{\boldsymbol{x}}}(t) = \tilde{\boldsymbol{A}}_{q(t)}\tilde{\boldsymbol{x}}(t) + \tilde{\boldsymbol{B}}_{q(t)}\boldsymbol{u}(t), \quad \tilde{\boldsymbol{x}}(t_0) = \boldsymbol{0}, \\ \tilde{\boldsymbol{y}}(t) = \tilde{\boldsymbol{C}}_{q(t)}\tilde{\boldsymbol{x}}(t), \end{cases} \tag{4}$$

with $\tilde{\boldsymbol{A}}_j \in \mathbb{R}^{r\times r}$, $\tilde{\boldsymbol{B}}_j \in \mathbb{R}^{r\times m}$, and $\tilde{\boldsymbol{C}}_j \in \mathbb{R}^{p\times r}$, and $r \ll n$. In many cases, see for instance [1], the reduced system matrices are obtained via Petrov–Galerkin projection, i.e., one constructs matrices $\boldsymbol{V}, \boldsymbol{W} \in \mathbb{R}^{n\times r}$ and then defines

$$\tilde{\boldsymbol{A}}_j := \boldsymbol{W}^{\mathsf{T}}\boldsymbol{A}_j\boldsymbol{V}, \qquad \tilde{\boldsymbol{B}}_j := \boldsymbol{W}^{\mathsf{T}}\boldsymbol{B}_j, \qquad \tilde{\boldsymbol{C}}_j := \boldsymbol{C}_j\boldsymbol{V}. \tag{5}$$

The goal of MOR is thus to derive in a computationally efficient and robust way the matrices $\boldsymbol{W}, \boldsymbol{V}$ such that the error $\boldsymbol{y} - \tilde{\boldsymbol{y}}$ is small in some given norm. One way to do so, originally presented in [6],

is to solve opportune defined GLEs to obtain the projection matrices $\boldsymbol{W}, \boldsymbol{V}$ and thus the reduced system (5). Therefore, solving efficiently large-scale generalized Lyapunov equation becomes crucial for MOR. More in detail the MOR algorithm from [6] proceeds in two steps. First, we have to define the matrices $\boldsymbol{A} := \boldsymbol{A}_1$ and $\boldsymbol{N}_j := \boldsymbol{A}_j - \boldsymbol{A}_1$ for $j = 1, \ldots, M$ and solve the GLEs

$$\boldsymbol{A}\mathcal{P} + \mathcal{P}\boldsymbol{A}^\mathsf{T} + \sum_{j=1}^{M} \left( \boldsymbol{N}_j \mathcal{P} \boldsymbol{N}_j^\mathsf{T} + \boldsymbol{B}_j \boldsymbol{B}_j^\mathsf{T} \right) = \mathbf{0}, \tag{6a}$$

$$\boldsymbol{A}^\mathsf{T}\mathcal{Q} + \mathcal{Q}\boldsymbol{A} + \sum_{j=1}^{M} \left( \boldsymbol{N}_j^\mathsf{T} \mathcal{Q} \boldsymbol{N}_j + \boldsymbol{C}_j^\mathsf{T} \boldsymbol{C}_j \right) = \mathbf{0}. \tag{6b}$$

Note that the matrix equations in (6) are of the form (1) by defining $\boldsymbol{B} := [\boldsymbol{B}_1, \ldots, \boldsymbol{B}_M]$ for (6a) and $\boldsymbol{B} := [\boldsymbol{C}_1^\mathsf{T}, \ldots, \boldsymbol{C}_M^\mathsf{T}]$, taking the transport on the other matrices for (6b). The symmetric and positive semi-definite solutions $\mathcal{P}, \mathcal{Q} \in \mathbb{R}^{n \times n}$ are referred to as the Gramians of (2). Second, let $\mathcal{P} = \boldsymbol{S}\boldsymbol{S}^\mathsf{T}$ and $\mathcal{Q} = \boldsymbol{R}\boldsymbol{R}^\mathsf{T}$ and compute the *singular value decomposition* (SVD) of the product of the Gramians factors

$$\boldsymbol{S}^\mathsf{T}\boldsymbol{R} = [\boldsymbol{U}_1, \boldsymbol{U}_2] \begin{bmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2 \end{bmatrix} [\boldsymbol{V}_1, \boldsymbol{V}_2]^\mathsf{T}, \tag{7}$$

and the projection matrices $\boldsymbol{V}$ and $\boldsymbol{W}$ are obtained via

$$\boldsymbol{V} = \boldsymbol{S}\boldsymbol{U}_1\boldsymbol{\Sigma}_1^{-1/2} \quad \text{and} \quad \boldsymbol{W} = \boldsymbol{R}\boldsymbol{V}_1\boldsymbol{\Sigma}_1^{-1/2}. \tag{8}$$

This procedure is denoted as square-root balanced truncation (see [1, Sec. 7.3]). The use of the solutions of (6) as system Gramians is justified by [6, Thm. 3], where the authors show that the image of $\mathcal{P}$ and $\mathcal{Q}$ encode the reachability set and observability set of the switched system (2).

**Main contributions:** To deal with the large-scale setting, we apply the stationary algorithm from [7] in combination with a subspace projection framework [8] to solve GLEs. We emphasize that this is a common strategy in the literature when dealing with GLEs. Our first contribution is the derivation of efficiently computable error estimates such that for any prescribed user tolerance `tol` an approximation $\tilde{\boldsymbol{X}}$ of (1) with guaranteed bound $\|\boldsymbol{X} - \tilde{\boldsymbol{X}}\|_2 \leq \texttt{tol}$ can be computed. Second, we show how the numerical error introduced in approximating (1) may deteriorate the quality and the stability of the ROM (4). This motivates us to propose a novel strategy that, by relying on the error certification provided by our algorithm, ensures stability and error certification of the MOR system. Finally, the results are validated through a synthetic example and a switched system arising from a parametric *partial differential equation* (PDE).

## References


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