Inverse Eigenvalue Difference Problems Arising in Quantum Sensing

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Abstract

This research is motivated by the emerging field of quantum sensing, which facilitates high-resolution sensing of gravitation, acoustic waves, and electromagnetic fields [1, 3, 4]. The discrete energy levels of coupled quantum dots (QDs) can be represented as eigenvalues of a quantum Hamiltonian matrix, whose entries are defined as polynomials of an applied electric field [2, 5]. Our aim is to recover the coefficients of these polynomials, which correspond to intrinsic physical constants, such as spin-coupling strength. The eigenvalue differences can be obtained from experimental measurements for varying electric field values. Standard inverse eigenvalue problems (IEP) seek to recover a constant matrix from the eigenvalues. In contrast, here the matrix elements are functions of a "tunable" parameter (the applied electric field) and only the *differences* between the eigenvalues are known. We formulate this as an inverse eigenvalue difference problem (IEDP).

**Problem formulation.** The steady-state energy levels of coupled QDs correspond to eigenvalues of quantum Hamiltonians. Specifically, the ground state of this system can be described by a $3 \times 3$ real symmetric matrix [2] of the form

$$G(F) = \begin{bmatrix} g_1(F) & y_0 & y_1 \\ y_0 & g_2(F) & y_2 \\ y_1 & y_2 & g_3(F) \end{bmatrix}, \tag{1}$$

where the diagonal elements depend quadratically on the applied electric field, $F \in \mathbb{R}$, as

$$g_i(F) = \alpha_i + \beta_i F + \gamma_i F^2, \quad i = 1, 2, 3 \ , \tag{2}$$

where the coefficients $\{\alpha_i, \beta_i, \gamma_i\}$ are real. We note that the assumption that the off-diagonal elements are independent of $F$ is a good approximation for weak electric fields and for weak tunnel coupling between the QDs. We shall further assume that $y_1 = y_0$ and $y_2 = 0$, which corresponds to symmetries between QDs.

Since $G(F)$ is symmetric, its eigenvalues are real. We denote its eigenvalues by $\{\xi_1(F), \xi_2(F), \xi_3(F)\}$. The physical measurements can be used to determine the *differences* between the eigenvalues of $G$. Thus, the measured data is provided over a set of $n$ values of $F \in \mathbb{R}$, denoted by $\{F_k\}_{k=1}^n$. The eigenvalue differences are denoted by

$$D_{2,1}(F) \equiv \xi_2(F) - \xi_1(F), \tag{3}$$
$$D_{3,1}(F) \equiv \xi_3(F) - \xi_1(F), \tag{4}$$
$$D_{3,2}(F) \equiv \xi_3(F) - \xi_2(F). \tag{5}$$

Note that $D_{3,2}(F) = D_{3,1}(F) - D_{2,1}(F)$. Hence, we consider the measured dataset to be

$$M = \left\{F_k, D_{2,1}(F_k), D_{3,1}(F_k)\right\}_{k=1}^n \ . \tag{6}$$

Our objective is to recover the coefficients that define $G(F)$. We can prove that, without loss of generality, one can set $g_1(F) = 0$. This has the effect of eliminating an arbitrary shift in the diagonal elements and in the eigenvalues of $G(F)$ that would satisfy the dataset $M$. We shall do so henceforth and redefine $G$ as

$$G(F) = \begin{bmatrix} 0 & y_0 & y_0 \\ y_0 & g_2(F) & 0 \\ y_0 & 0 & g_3(F) \end{bmatrix} . \tag{7}$$

We denote the vector of coefficients as $\mathbf{p} = [y_0, \alpha_2, \beta_2, \gamma_2, \alpha_3, \beta_3, \gamma_3] \in \mathbb{R}^7$. In this work, we seek to solve the following:

**Inverse Eigenvalue Difference Problem (IEDP)**

Given the eigenvalue difference dataset $M$ in (6), find coefficients $\mathbf{p}$, such that $G(F)$ in (7) generates $M$.

In particular, our work addresses two inter-related questions:

1. What optimization approach is efficient for solving this IEDP, especially in the presence of noisy data?
2. What domain knowledge can we utilize to improve the efficacy of the proposed approach?

## References


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