

A Time-Frequency Method for Acoustic Scattering in Unfriendly Domains

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Abstract

The acoustic scattering problem asks for the recovery of a scattered wavefield that is produced when an incoming incident wavefield strikes a scattering object. If sound-soft boundary conditions are imposed and the incident wavefield u_{inc} is pulse-like and away from the scatterer at $t = 0$, then the scattered field u satisfies the following:

$$\frac{\partial^2 u}{\partial t^2}(x, t) - c^2 \Delta u(x, t) = 0, \quad (x, t) \in \Omega \times [0, T], \quad (1)$$

$$u(x, 0) = \frac{\partial u}{\partial t}(x, 0) = 0, \quad x \in \Omega, \quad (2)$$

$$u(x, t) = -u_{inc}(x, t), \quad (x, t) \in \partial\Omega \times [0, T]. \quad (3)$$

Here, c is the wave speed associated with the domain, Ω is an exterior domain, and $[0, T]$ is some relevant time period over which the scattered waves are observable. This problem is challenging to solve for a number of reasons. Domain discretization schemes must manage the fact that the exterior domain is unbounded as $|x| \rightarrow \infty$ and impose artificial absorbing boundary layers that appropriately handle outgoing waves. Traditional direct-in-time methods must combat pervasive issues related to accumulating dispersive error and potentially prohibitive restrictions on time step sizes. When the domain includes corners, cusps, or trapping regions where the scattered waves can get stuck and decay very slowly, these issues become severely exacerbated.

Under the condition that the incident wavefield is enveloped by a Gaussian or is otherwise approximately bandlimited, some of the complications of direct-in-time methods can be avoided by using *hybrid time-frequency solvers* [1, 8]. This class of methods considers an equivalent problem in the frequency domain. If $\hat{U}(x, \omega)$ is the Fourier transform of $u(x, t)$ with respect to time, then it follows that \hat{U} satisfies the Helmholtz equation with a continuously parametrized wavenumber:

$$\Delta \hat{U}(x, \omega) + \frac{\omega^2}{c^2} \hat{U}(x, \omega) = 0, \quad x \in \Omega, \quad (4)$$

$$\hat{U}(x, \omega) = -\hat{U}_{inc}(x, \omega), \quad x \in \partial\Omega, \quad (5)$$

The Sommerfeld radiation condition is additionally imposed at all ω . Now $u(x, t)$ can be expressed in terms of an inverse Fourier transform. If $\hat{U}(x, \omega)$ is sufficiently small for all x whenever ω is outside the band $[W_1, W_2]$, then

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{U}(x, \omega) e^{-i\omega t} d\omega \approx \frac{1}{2\pi} \int_{W_1}^{W_2} \hat{U}(x, \omega) e^{-i\omega t} d\omega. \quad (6)$$

The primary task in these methods is the evaluation of the integral in (6) via quadrature, which in turn requires the solving of the Helmholtz equation at quadrature points $\{\omega_1, \omega_2, \dots, \omega_N\}$, and then the evaluation of the solutions over all domain points of interest. There are major advantages to this formulation, including the ability to compute solutions that are virtually free of dispersive error, and the ability to evaluate $u(x, t)$ at arbitrary points in time (time-skipping). Moreover, solutions to the Helmholtz equation can be expressed via boundary integral equations that nicely handle the exterior domain by reducing it to a contour integral around the boundary of the scatterer [7].

However, there is no free lunch! These methods become prohibitively expensive or intractable when the domain involves corners, trapping regions, or both. We introduce developments that overcome these challenges so that hybrid time-frequency methods are effective in domains with these “unfriendly” features. Example domains where our methods work well include multiply-connected regions (e.g., whisper galleries or panel sets), as well as keyhole regions with severe trapping, multiple corners, and long channels.

Our work involves two major developments that make hybrid time-frequency methods effective in unfriendly domains:

For trapping regions: A damping+correction fast quadrature scheme for the inverse Fourier transform that combines contour integration with fast transforms (the FFT and the fast sinc transform) to make evaluation of the integral in (6) tractable even when there is trapping.

For domains with corners: State-of-the-art hierarchical linear algebra routines based on recursive skeletonization and the balanced use of “universal skeletons” for solving and evaluating the broadband Helmholtz equation. Excellent efficiency makes it possible to handle discretizations with many points (e.g., induced by refinement into corners).

We briefly discuss these two developments, starting with a method to make evaluation feasible in domains with trapping.

There are two reasons that the integral in (6) can be challenging to discretize efficiently. First, if t is large, the integral becomes highly oscillatory and naive discretizations (e.g., Gauss-Legendre quadrature) require many points to capture the oscillations. Coupled to this problem is the fact that when $u(x, t)$ is in a trapping region and thus decays slowly over time, $\hat{U}(x, \omega)$ has poles in the complex plane very near to the interval $[W_1, W_2]$. These poles are associated with near-resonant modes of the Helmholtz operator. Since $\hat{U}(x, \omega)$ has only a very small region of analyticity that encloses $[W_1, W_2]$, it is inherently difficult to approximate with polynomials.

To improve upon the analyticity properties, we note that the poles of $\hat{U}(x, \omega)$ always lie below the real line. This suggests the consideration of the perturbed function $\hat{U}(x, \omega + i\delta)$, where $\delta > 0$. One can view this perturbation as the introduction of mild damping into the solution. Since $\hat{U}(x, z)$ is analytic with respect to z in the upper half plane, we can apply Cauchy’s integral theorem on a rectangular contour in the upper half plane and represent the undamped solution as

$$\int_{W_1}^{W_2} \hat{U}(x, \omega) e^{-i\omega t} d\omega = - \underbrace{\int_{W_1}^{W_2} \hat{U}(x, \omega + \delta i) e^{-i(\omega + \delta i)t} d\omega}_{I_\delta} - I_{cL} - I_{cR}, \quad (7)$$

where the correction terms I_{cL} and I_{cR} are integrals along the vertical sides of the rectangle. There is an inherent upper limit on δ . We show that if it is taken too large, the correction terms cannot be stably evaluated. However, the room afforded by δ is generous enough that when paired with a fast evaluation scheme for the integral I_δ , we are able to solve the acoustic scattering problem even with severe trapping over long time horizons.

The integral I_δ is still problematic in that it can be highly oscillatory when t is large. To manage this, we follow an idea originally presented in [1] that is also related to sampling schemes for bandlimited functions [6]. We approximate the periodization of $\hat{U}(x, \omega + \delta i)$ with a trigonometric polynomial. Critically, this is possible because of the improved analyticity afforded by damping. Substituting this approximation into the integral I_δ results in a simplification of the integral so

that for each fixed x ,

$$I_\delta \approx \frac{W_2 - W_1}{2\pi(2m + 1)} e^{-it((W_2 - W_1)/2 + W_1 + i\delta)} \sum_{j=-m}^m (-1)^j c_j \operatorname{sinc} \left(\frac{(W_2 - W_1)t}{2\pi} - j \right), \quad (8)$$

where $\operatorname{sinc}(x) = \sin(\pi x)/(\pi x)$. The set of coefficients $\{c_j\}_{j=-m}^m$ is x -dependent and can be found using the FFT on equally spaced samples of $\hat{U}(x, \omega + \delta i)$ over the band $[W_1, W_2]$. Rather than naively evaluating the sum, we apply the fast sinc transform [5] (via the nonuniform FFT of type-III [2]) to evaluate the weighted sum of sines at locations $\{t_1, t_2, \dots, t_N\}$ in only $\mathcal{O}(m \log m + N)$ operations. Note that the complexity of this method no longer depends linearly on t , as would be true with a naive discretization. Instead, it depends on m , which has an implicit but much weaker dependence on t since the size of t for which $u(x, t)$ is relevant is correlated to the distance of $[W_1, W_2] + \delta i$ from the poles associated with near-resonant modes.

Even with damping, the discretization of I_δ still requires $2m + 1$ solves of the Helmholtz equation, and then evaluations of each of these solutions at every point x of interest in the domain. Corners induced by singularities in the solutions require special care. They can be handled with highly specialized quadrature that requires precise knowledge about the geometry [3, 9], or they can be handled with refinement strategies in the Nyström discretization of the integral. The latter is practical, but makes the solve and evaluation steps much more expensive. To handle the expense, we employ so-called *universal skeletons* in a fast solver based on recursive skeletonization. Evaluations are then handled with the fast multipole method. Universal skeletons were first introduced in [4]. They supply a way to reuse low rank approximation factors in hierarchical discretizations of boundary integral equations as the wavenumber parameter changes. We discuss how they can be adapted and applied in balanced and effective ways in the context of the acoustic scattering problem.

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